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On the Homology Theory of Operator Algebras

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Abstract

For given Banach algebras A and B, we define a free resolution of algebra$B$ over a homomorphism $f : A \to B$ and study some properties of the relative cyclic homology.

1 Introduction.

Given Banach algebrasA, B, let $f : A \to B$ be algebras homomorphism. We define a free resolution of algebra B over the homomorphism $f : A \xrightarrow{i} R \xrightarrow{\pi} B$, where i is an inclusion and π is a quasi-isomorphism, and use this fact to define the relative cyclic homology

$$HC_*(A \xrightarrow{f} B) = H_*(R/(A + [R, R] + (1 - t_n)) \tag{1}$$

where$[R, R]$ is the commutant of algebra R, and study its main properties.

First, we recall some definitions and facts from [2, 4, 6]. Let A be a unital Banach algebra over a commutative ring K ($K = R$ or C). Define the complex $C(A) = (C_*(A), b_*)$, where$C_n(A) = A \otimes ... \otimes A$ is the tensor product of algebra $(n + 1 times)$

and , $b_* : C_n(A) C_{n-1}(A)$ is the boundary operator

$$b_n(a_0 \otimes ... \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_n \otimes ... \otimes a_i a_{i-1} \otimes ... \otimes a_{n-1} \tag{2}$$

We can easily verify that $b^{n+1}b^n = 0$ and hence$Ker\ b_n \supset \operatorname{Im} b_{n-1}$.The group $H_n(A) = H(C(A)) = \frac{Kerb_n}{\operatorname{Im} b_{n-1}}$is called the simplicial (hochschild) homology of algebra A.

Note that $Ker\ b_n$ is always closed, but $\operatorname{Im} b_{n-1}$, in general , is not closed.

Considering a unital Banach algebra A, one acts on the complex $C(A)$, by the cyclic group of order $n+1$ by means of the operator $t_n : C_n(A) \to C_n(A)$ such that :

$$t_n(a_0 \otimes ... \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes ... \otimes a_{n-1} \tag{3}$$

The quotient complex $CC_n(A) = C_n(A)/\operatorname{Im}(1 - t_n)$ is a subcomplex of a complex $C_n(A)$. Following [Hel, wood 4] the cyclic homology of algebra$A$ is the homology of the complex $CC_*(A)$.

2 Free resolution. [[1]].

In this part, we discuss the existenc of the free algebra resolution. Let $V = \sum_{n-0}^{\infty} V_n$ be a graded vector space over ring F ($F = R$ *or* C). Suppose that R is a differential graded $F - algebra$ and let $R\langle V\rangle = R^* T_K(V)$ be the free product of algebras, where $T_K(V) = \sum_{j \geq 0} V^{\otimes j}$ is the tensor algebra over F. The product in $R\langle V\rangle$ is given by

$$(r_1 e_1 ... r_n e_n r_{n+1}).(\hat{r}_1 \hat{e}_1\ ...\ \hat{r}_k \hat{e}_k \hat{r}_{k+1}) \tag{4}$$

$$= (r_1 e_1 ... r_n e_n (r_{n+1}\ \hat{r}_1)\ \hat{e}_1\ ...\ \hat{r}_k \hat{e}_k \hat{r}_{k+1}) r_i, \quad \hat{r}_j \in R, e_i\ \hat{e}_j \in T_K(V))$$

Definition 1 *1.1. Let $f : R_1 \to R_2$ be a homomorphism of differential graded $F - algebras$. An algebra R_2 is a free algebra over the homomorphism f if there exists an isomorphism$\alpha : R_1\langle V\rangle \approx R_2$, where E is a differential graded vector space with the following commutative diagram:*

$$\begin{array}{ccc} R_1 & \xrightarrow{f} & R_2 \\ & i \searrow & \downarrow \alpha \\ & & R_1\langle E\rangle \end{array} \tag{5}$$

where i is inclusion map.

Lemma 1 *1.2. Let $f : A \to B$ be a homomorphism of $F - algebras$. Then there exists a differential graded algebra $R = \sum_{i=0} R_i$with the following properties:*

(i) $\pi$is surjection and the following diagram is commutative

$$\begin{array}{ccc} & & R \\ & i \nearrow & \downarrow \pi \\ A & \underset{f}{\rightarrow} & B \end{array} \tag{6}$$

where i is the inclusion map.

Clearly there is an isomorphism $j : R \to A$ such that$j^0 i = 1_A$.

(ii)π is quasi-isomorphism, $i.e. \pi_* : H_*(A) \to H_*(B) = B$, where B is a differential graded algebra,

$$(B)_i = \begin{cases} B, & i = 0 \\ 0, & i \succ 0 \quad and\ the\ differetial\ \partial^B = 0 \end{cases} \tag{7}$$

(iii) The differential graded algebra R is free over the homomorphism $i : A \to R$.

Definition 2 *1.3. The differential graded algebra which satisfies the conditions* $(i), (ii)$ *and* (iii) *of lemma 1.2 is called a free resolution of algebra* B *over*f.

Proof of lemma 1.2.

First step. We construct a commutative diagram of algebras

$$\begin{array}{ccc} & & R^{(0)} \\ & i_0 \nearrow & \downarrow \pi_0 \\ A & \underset{f}{\rightarrow} & B \end{array} \tag{8}$$

where $R^{(0)}$ is free over the homomorphism $i^0 : A \to R^{(0)}, \pi_0$an surjection. Define $A\left\langle (t_i) \right\rangle = A\left\langle (t_i) \right\rangle$, where $E(t_i)$is an involutive vector space generated by $\{t_i\}$, or generated by the family $\{t_i t_i^*\}$. The automorphism $* : E(t_i) \to E(t_i)$ is given as follows $^*(t_i) = (t_i^*) * (t_i^*) = t_i$. We choose a system $\left\{\Re_i^{(0)}\right\}$of generators in algebra B. This family is assumed to be closed under an involutive on$B$.Now, let $R^{(0)} = A\left\langle (t_i^{(0)}) \right\rangle$, where $t_i^{(0)}$ is equivalent to the generator $\left\{\Re^{(0)}\right\}$ in algebra$B$, and suppose that $\beta_i^{(0)} = t_i^{(0)} or (t_i^{(0)})^*$. We define $\pi_0$using the universal property of $R^{(0)}$. Let π_0 be the unique involutive algebras $R^{(0)} \to B$ homomorphism, which restricts$f$ on$A$ and sends $t_i^{(0)} to \Re_i^{(0)}$.Since $i_0 : A \to A\left\langle t_i^{(0)} \right\rangle$ is an inclusion map, $i_0(a) = a, i_0$ is an algebras homomorphism and $\pi_0 i_0(a) = \pi_0(a) = f(a)$. Hence, diagram (8) is commutative and π_0

is surjective.Let$j_0 : R^{(0)} \to A$ be the unique algebras homomorphism restricting to the identity on A and mapping $t_i^{(0)} to\ zero.R^{(0)}$ is a differential graded $K - algebra$;

$$(R^{(0)})_i = \begin{cases} R^{(0)} & i = 0 \\ 0, & i \succ 0 \quad and\ the\ differetial\ \partial^{R^{(0)}} B_i^{(0)} = 0 \end{cases} \tag{9}$$

The algebra $R^{(0)}$is free over the homomorphism $i_0 : A \to R^{(0)} since R^{(0)} = A\left\langle t_i^{(0)} \right\rangle$.

Second step. We construct the second commutative diagram

$$\begin{array}{ccc} & & R^{(1)} \\ & i_1 \nearrow & \downarrow \pi_1 \\ A & \underset{f}{\to} & B \end{array} \tag{10}$$

where $R^{(1)}$ is free over the homomorphism $I_1 = A \to R^{(1)}$ and π_1 surjection. Choose a system $\left\{\Re_j^{(1)}\right\}$ of generators of $Ker\ \pi_0$, which is closed under involution. Let $t_i^{(1)}$ be indeterminate which are bijection with the $\Re_j^{(1)}$. Define$R^{(1)} = A\left\langle t_i^{(0)}, t_j^{(1)} \right\rangle$, where $t_i^{(0)}$ is defined above. Suppose that $\beta_j^{(1)}$ denotes $t_i^{(1)} or\ (t_i^{(1)})^*$. The homomorphism π_1 is defined to be the unique algebras $R^{(1)} \to B$ restricting to $\pi_0\ on\ R^{(0)}$ and sending $t_i^{(1)}\ to\ zero$. We can see, from the above discussion, that the homomorphism π_1 can be defined as $\pi_0$and that π_1 is surjective since $\pi_1(\ \beta_j^{(0)}) = \Re_i$, $\pi_1(\ \beta_j^{(1)}) = 0_i$. The homomorphism $i_1 : A \to A\left\langle t_i^{(0)}, t_j^{(1)} \right\rangle$ is inclusion. The diagram (10) is commutative since $(\pi_1 i_1)(a) = \pi_1(a) = f(a)$.

The homomrphism j_1 is defined to be the unique homomorphism: $R^{(1)} \to A$ of involutive algebras restricting to identity on A and mapping $t_i^{(1)}\ to\ zero$. The algebra $R^{(1)} = A\left\langle t_i^{(0)}, t_j^{(1)} \right\rangle$is free over i.Finally, we have a differential graded algebra

$$R^{(1)} = (R^{(1)})_0 \oplus (R^{(1)})_1 \oplus ..., \deg \beta_i^{(1)} = 0, \deg \beta_j^{(1)} = 1. \tag{11}$$

The differential $\partial_i^{R^{(1)}}$is the unique derivation on $R^{(1)}$ satisfying the graded Leibntiz rule and commuting with the involution which restricts to zero on$R^{(1)}$ and sends $t_i^{(1)} to \Re_j^{(1)}$.So

$$\partial_i^{R^{(1)}} \beta_i^{(0)} = 0, \partial_i^{R^{(1)}} \beta_i^{(0)} = \Re_j^{(1)} \in Ker\pi_0, i \succ 1.$$

In the same manner, we can consider the commutative diagram

$$\begin{array}{ccc} & & R^{(2)} \\ & i_2 \nearrow & \downarrow \pi_2 \\ A & \underset{f}{\rightarrow} & B \end{array} \tag{12}$$

where$R^{(2)} R^{(2)} = A\left\langle t_i^{(0)}, t_j^{(1)}, t_k^{(2)} \right\rangle$ is a differential graded algebra,

$$R^{(2)} = (R^{(2)})_0 \oplus (R^{(2)})_1 \oplus (R^{(2)})_2 \oplus \deg \beta_i^{(0)} = 0, \deg \beta_j^{(0)} = 1, \deg \beta_j^{(0)} = 2 \tag{13}$$

The differential algebra $R^{(2)}$ is also defined by using a universal property and, hence,

$$\partial_i^{R^{(2)}} \beta_i^{(0)} = 0, \partial_1^{R^{(2)}} \beta_i^{(0)} = \Re_j^{(1)}, \partial_1^{R^{(2)}} \beta_j^{(2)} = \Im_k^{(2)} = 0, i \succ 2. \tag{14}$$

Consequently, we can construct an involutive algebra $R^{(i)}, i \geq 0$ with the following commutative diagram:

$$\begin{array}{cccccccccc} & R^{(0)} & \overset{p_0}{\rightarrow} & R^{(1)} & \overset{p_1}{\rightarrow} & ...R^{(n-1)} & \overset{p_{n-1}}{\rightarrow} & R^{(n)} & \overset{p_n}{\rightarrow} \\ i_0 \nearrow & \downarrow \pi_0 & & \downarrow \pi_1 & & \downarrow \pi_{n-1} & & \downarrow \pi_n & \\ A \rightarrow & B & === & B & === ... & B & === & B & === \end{array} \tag{15}$$

Where π_i is surjection, $i \geq 0, i_n = P_{n-1} o ... o P_0 o i_o$ is an inclusion map from $A \ to \ R^{(n)}, P_i$ is also is an inclusion map from

$$p_i : A\left\langle t_{m_0}^{(0)}, t_{m1}^{(1)}, ..., t_{m_i}^{(i)} \right\rangle \qquad to \qquad A\left\langle t_{m_0}^{(0)}, t_{m1}^{(1)}, ..., t_{m_i}^{(i)}, t_{m_{i+1}}^{(i+1)} \right\rangle \tag{16}$$

Define $i_n = q_{n-1} o ... o q_0 o j_o$, where q_n is the projection of P_i.

The diagram 15 is commutative since $i_{n+1}(\beta_i^{(n)}) = \pi_n(\beta_i^{(n)}) = 0, \qquad n \geq 0$

Define $R = \lim R_n, \pi = \lim \pi_n, i = \lim i_n, j = \lim j_n$ .Then the differential graded algebra R satisfies the items of lemma 1.2 since:

(1) $\pi = \lim \pi_n$ is surjection, the diagram

$$\begin{array}{ccc} & & R^{(2)} \\ & i_2 \nearrow & \downarrow \pi_2 \\ A & \underset{f}{\rightarrow} & B \end{array} \tag{17}$$

is commutative since$i(a) = a$, $\pi(a) = f(a)$.

(2)π is quasi-isomorphism of differential graded algebras

$$\begin{array}{cccccccccc} (R)_0 & \underset{\longleftarrow}{\partial_0^R} & (R)_1 & \underset{\longleftarrow}{\partial_1^R} & ... & \underset{\longleftarrow}{\partial_n^R} & (R)_n & \underset{\longleftarrow}{\partial_{n+1}^R} & ... \\ \downarrow \pi_1 & & \downarrow \pi_2 & & & & \downarrow \pi_n & & \\ B & \longleftarrow & 0 & \longleftarrow & ... & \longleftarrow & 0 & \longleftarrow & \cdots \end{array} \tag{18}$$

where $\partial_i^R = \lim \partial_i^R$, $(R)_0 = \ker\ (\pi)_0 = B, \operatorname{Im}\partial^R = \ker \partial_0^R$, *i.e.* $H_0(R) = H_i(R) = 0$.

(3) The differential graded algebras$R$ is free over the homomorphism$i$: $A \to R$,since $R = E$, E is a vector space generated by the system:

$$\left\{t_{i_1}^{(0)}, t_{i_1}^{(1)}, ...t_{i_1}^{(n)}\right\} \tag{19}$$

3 The cyclic homology.

In this part, we define the relative cyclic homology and study its properties. Let f be a homomorphism of Banach algebras A and B over a field K (K is real or complex number set). Let R_f^B be a free resolutionof algebra B over f and, for $r_1, r_2 \in R_f^B$, let

$[r_1, r_2] = r_1, r_2 - (-1)^{|r_1||r_2|} r_1, r_2$ where $|r_i| = deg\ r_i, i = 1, 2$.Let$C = [R_f^B, R_f^B]$ be the linear space generated by $[r_1, r_2], r_1, r_2 \in R_f^B$. It is clear that t$C = [R_f^B, R_f^B]$is k-submodule of k-module. We construct the complex

$C = [R_f^B, R_f^B]$. Clearly, from the definition of R_f^B , that Im $(1 - t_n)$ is a subcomplex of R_f^B . We have

$$\begin{aligned}
\partial [r_1 r_2] &= r_1 r_2 - (-1)^{|r_1||r_2|} r_1 r_2 \qquad (20)\\
&= \partial r_1 r_2 + (-1)^{|r_2|} r_1 \partial r_2 - (-1)^{|r_1||r_2|}(\partial r_2 r_1 + -(-1)^{|r_2|} r_2 \partial r_1)\\
&= \partial r_1 r_2 - (-1)^{|r_2|(|r_1|+1)} r_1 \partial r_2 + (-1)^{|r_1|}(r_1 \partial r_2 - (-1)^{|r_1|(|r_2|+1)} \partial r_2 r_1)\\
&= [\partial r_1 r_2] + (-1)^{|r_1|} [r_1 \partial r_2], \qquad |\partial r_i| = |r_1| - 1 \qquad i = 1, 2
\end{aligned}$$

Then $[R_f^B, R_f^B]$ is subcomplex in R_f^B . Therefore , the chain complex of $K - module$ $([R_f^B, R_f^B])$ is a subcmplex of R_f^B.

Definition 3 *2.1. Let $f : A \to B$be F-algebras $(charF = 0)$ homomorphism, R_f^B be a free resolution of algebra B over f. Then the relative cyclic homology is defined as follows:*

$$HC_*(A \xrightarrow{f} B) = H_*\left(\frac{R_f^B}{(A + [R, R] + \operatorname{Im}(1 - t_n)}\right) \tag{21}$$

The main properties of the relative cyclic homology are submitted in Theorem 2.2, 2.6, 2.7.

Theorem 2.2. Let A be an algebra. Then $HC_i(A \to B) = HC_{i-1}(A)$, where $HC_{i-1}(A)$ is the cyclic homology of F-algebras $(charF = 0)$.

Proof. To do this, we need the following definition and lemmas.

Definition 4 *2.3. The F -algebra $A\langle t\rangle$ generated by the elements $a_0ta_1t...ta_n$, $n\geq 0$,can be considered as differential graded algebras by requiring that the morphism $A\to A\langle t\rangle$ is a morphism of differential graded algebras (A is viewed as a differential graded algebra concentrated indegree 0) and the $degt=1$, $\partial t=0$ and $t^*=t$.*

Lemma 2 *2.4. The algebra $A\langle t\rangle$ is splitable. It is free algebra resolution of the algebra$B=0$over the homomorphism $A\to 0$.*

Proof. Define the following chain complex

$$A\xleftarrow{\partial}AtA\xleftarrow{\partial}AtAt\xleftarrow{\partial}...\xleftarrow{\partial}At...tA\xleftarrow{\partial}... \tag{22}$$

where $At...tA$ (n-times) is a K-module and the boundary operator ∂ is given by

$$\begin{aligned}\partial(a_0ta_1t...ta_{n-1}ta_n) &= \sum_{i=0}^{n-1}(-1)^i a_0ta_1t...ta_i(\partial t)a_{i+1}t...ta_n \\ &= \sum_{i=0}^{n-1}(-1)^i a_0ta_1t...t(a_ia_{i+1})t...ta_n.\end{aligned} \tag{23}$$

Note that the differential ∂ in $A\langle t\rangle$ is equivalent to the operator δ'_n : $C_n(A)\to C_{n-1}(A)$(see [[4]]), defined by

$$\delta'_n(a_0\otimes...\otimes a_n)=\sum_{i=0}^{n-1}(-1)^i a_0\otimes a_1\otimes...\otimes a_ia_{i+1}\otimes...\otimes a_n.$$

Following [[4]], the complex $(C_n(A),\delta'_n)$ is splitable and so the complex $A\langle t\rangle$ is also splitable, that is, $H_*(A\langle t\rangle)=0$. Therefore, algebra $A\langle t\rangle$ is free resolution of the algebra $B=0$ over the homomorphism $A\to 0$.

Lemma 3 *2.5. The complex $A\langle t\rangle/[A,A\langle t\rangle]$ is standard simplicial (Hochschild) complex.*

Proof. Consider the factor complex $A\langle t\rangle/[A,A\langle t\rangle]$. It's generated by the elements $a_0ta_1t...ta_{n-1}t$, since

$$a_0ta_1t...ta_{n-1}ta_n=a_na_0\times ta_1t...ta_{n-1}t(\mathrm{mod}\,[A,A\langle t\rangle]). \tag{24}$$

The action of the differential ∂ on the complex $A\langle t\rangle / [A, A\langle t\rangle]$ is given by;

$$\begin{aligned}\partial(a_0ta_1t...ta_{n-1}ta_n) &= \sum_{i=0}^{n-1}(-1)^i a_0ta_1t...t(a_ia_{i+1})t...ta_n \\ &+(-1)^n a_na_0ta_1t...a_{a-1}t.\end{aligned} \tag{25}$$

Consider the complex

$$A \xleftarrow{id} A \xleftarrow{\delta} A^{\otimes 2} \xleftarrow{\delta} ... \xleftarrow{\delta} A^{\otimes n} \xleftarrow{\delta} ..., \tag{26}$$

where δ is the differential in the standard Hochschild complex ([4]). Since the space $(A\langle t\rangle / [A, A\langle t\rangle])_{n+1}$ identifies with the space

$$A^{\otimes n+1} : a_0ta_1...ta_nt \to a_0 \otimes a_1 \otimes ... \otimes a_n \tag{27}$$

and the differential in $A\langle t\rangle / [A, A\langle t\rangle]$ identifies with the differential in the standard Hochschild complex, then the complex $A\langle t\rangle / [A, A\langle t\rangle]$ is the Hochschild (simplicial) complex .

Now ,we prove Theorem 2.2.

Consider the factor complex : $A\langle t\rangle / [A, A\langle t\rangle] + \mathrm{Im}(1 - r_n)$, such that

$$a_0ta_1t...ta_{n-1}t = (-1)^n a_na_0t...a_{a-1}t \tag{28}$$

where deg , $\deg a_0ta_1t...ta_{n-1}t = 0$ $\deg a_0ta_1t...ta_nt = n + 1$.

The cyclic homology of $A\langle t\rangle$ is the homology of the complex $A\langle t\rangle / [A\langle t\rangle,\ A\langle t\rangle]+$ $\mathrm{Im}(1 - t_n)$. By factoring $A\langle t\rangle$, first by the subcomplex $A \longleftarrow 0 \longleftarrow 0 \longleftarrow ...$ and second by the subcomplex $[A\langle t\rangle,\ A\langle t\rangle] + \mathrm{Im}(1 - t_n)$ we get a homomorphism $CC_*(A \to 0) \to CC_{*-1}(A)$, which induces an isomorphism of the cyclic one homology groups $HC_*(A \to 0) \to HC_{*-1}(A)$.

Theorem 4 *2.6:$f : A \to B$ be a homomorphism of algebras over a field$K(char K =$ $0)$. Then the relative cyclic homology$HC_i(A \xrightarrow{f} B)$ does not depends on the choice of the resolution.*

Proof. The homomorphism f induces homomorphism of chain complexes

$$f_* : CC_*(A) \to CC_*(B) \tag{29}$$

Where $CC_*(A)$ is a cyclic complex. Consider the diagram

$$\begin{array}{ccc} & & R_f^B \\ & i \nearrow & \downarrow \pi \\ A & \xrightarrow[f]{} & B \end{array} \tag{30}$$

Where R_f^B is defined above, i is an inclusion map. The idea of proof is to show that the cone of the map i is quasi-isomorphic to an arbitrary category ([5]), to the complex $R_f^B / \left[R_f^B, R_f^B\right] + \mathrm{Im}(1-t_n)$, Since

$$H_i(R_f^B) = \begin{cases} B, & i=0 \\ 0, & i \succ 0 \end{cases} \tag{31}$$

Then the isomorphismπ_* $CC_*(R_f^B) \to CC_*(B)$ induces an isomorphism of the homology of these complexes. Since i_* $CC_*(A) \to CC_*(R_f^B)$ is an inclusion, then

$$HC_i(A \xrightarrow{f} B) \to HC_i(A \xrightarrow{gof} C) \to HC_i(A \xrightarrow{g} C) \to HC_{i-1}(A \xrightarrow{f} B) \to ... \tag{32}$$

$M(i_*) \approx CC_*(R_f^B) \diagup CC_*(A)$, where $M(i_*)$ is a cone of i (see[5]).

Note that, the symbol $\approx$ denotes a quasi-isomorphism. It is clear, from the above discussion, that the following diagram is commutative:

$$\begin{array}{ccc} & & CC_*(R_f^B) \\ & i_* \nearrow & \downarrow \pi_* \\ CC_*(A) & \underset{f}{\longrightarrow} & CC_*(B) \end{array} \tag{33}$$

And hence $M(f_*) \approx CC_*(R_f^B) \diagup CC_*(A)$. Following [[1]], we have $CC_*(R_f^B) \diagup CC_*(A) \approx R_f^B \diagup A + [R_f^B, R_f^B]$, where CC_* is the Connes cyclic complex, and by using the spectral sequence $E_{ij}^2 =^{\epsilon} H_*(z \diagup 2, H_*(R_f^B)) =^{\epsilon} H_{i+j}(R_f^B)$, we have

$$CC_*(R_f^B) \diagup CC_*(A) \approx R_f^B \diagup A + [R_f^B, R_f^B] + \mathrm{Im}(1-t_n) \tag{34}$$

So,$M(F_*) \approx R_f^B / A + [R_f^B, R_f^B] + \mathrm{Im}(1-t_n)$. Then $HC_i(A \xrightarrow{f} B)$ does not depend on the choice of R_f^B .

Theorem 5 *2.7. Let A, B and D be involutive algebra. Then the following sequence $A \xrightarrow{f} B \xrightarrow{g} D$ induces the long exact sequence of relative cyclic homology*

$$HC_i(A \xrightarrow{f} B) \to HC_i(A \xrightarrow{gof} D) \to HC_i(A \xrightarrow{g} D) \to HC_{i-1}(A \xrightarrow{f} B) \to ... \tag{35}$$

Proof. In Theorem 2.6, it has been proved that any homomorphism f : $A \to B$ of involutive algebra in an arbitrary category is equivalent to an inclusion $i : A \to R_f^B$. Then, for a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of involutive algebra, we have the following complex

$$\begin{array}{ccccccc} A & \xrightarrow{i} & R_f^B & === & B & \xrightarrow{i} & R_f^D \\ f \searrow & & \| & & f \searrow & & \| \\ & & B & & & & D \end{array} \tag{36}$$

Consider the following sequence of mapping cones

$$0 \to M(i_*) \to M(i'_*) \to M(i_* o i'_*) \to 0 \tag{37}$$

In general, the sequence (37) is not exact. The composition of two morphism will be zero. However , the cone over the morphism $M(i_*) \to M(i'_*)$ is canonically homotopy equivalent to $M(i_* o i'_*)$. So we get the following exact sequence of the relative cyclicl homology

$$HC_i(A \xrightarrow{f} B) \to HC_i(A \xrightarrow{gof} C) \to HC_i(A \xrightarrow{g} C) \to HC_{i-1}(A \xrightarrow{f} B) \to ... \tag{38}$$

In the following we give an example of the cyclic homology of tensor algebra by using the free resolution fact. Let$A$ be $F - algebra, (charF = 0)$ and M is A- bimodule. For a chain complex $V_\bullet$ of modules, consider the complex$S^n(A, V_\bullet) = A \otimes_{A \otimes A^{op}} V_\bullet^{\otimes(k+1)}$. If we act on $S^n(A, V_\bullet)$ by the cyclic group Z_{n+1} of order $(n+1)$ by means of automorphisms

$$t_n(v_0 \otimes ... \otimes v_n) = (-1)^\mu v_n \otimes v_0 \otimes ... \otimes v_{n-1} \tag{39}$$

where $\mu = (\deg\, p_n)(\sum_{i=0}^{n-1} \deg\, p_i)$.

If $V_\bullet$ is a free resolution of $A - bimodule$ M, then the complex $S^n(A, V_\bullet)$ can be considered by as complex $S^n(A, M)$.

Example 1 *2.8. Let, M be $A-bimodule$, where A is $K-algebra$ ($charK = 0$), is a tensor algebra and $Tor_i^A(M, M) = 0$, $i \succ 0$, then*

$$HC_i(T_A(M)) = HC_i(A) \oplus (\oplus_{n=0}^{\infty} H_i(Z_{n+1}; S^n(A, M))) \tag{40}$$

Proof. Suppose $V_\bullet$ is a free resolution of $A-bimodule$ M, then. According the condition $Tor_i^A(M, M) = 0$, $i \succ 0$, the space $T_A(A)$ is a free resolution of algebra $T_A(M)$ over inclusion $i : A \to T_A(M)$. Using theorem 2.7. the long exact sequence of relative cyclic homology of the following sequence $A \xrightarrow{i} T_A(M) \longrightarrow 0$, is given by

$$\begin{array}{ll} ... \longrightarrow & HC_i(A \xrightarrow{i} T_A(M)) \longrightarrow HC_i(A \xrightarrow{0} 0) \\ \longrightarrow & HC_i(T_A(M) \longrightarrow 0) HC_{i-1}(A \xrightarrow{i} T_A(M)) \xrightarrow{0} ... \end{array} \tag{41}$$

Since$A$ is a direct sum of $T_A(M)$, we have

$$0 \longrightarrow HC_i(A \xrightarrow{i} T_A(M) \longrightarrow HC_i(A) \longrightarrow HC_i(T_A(M)) \longrightarrow 0 \tag{42}$$

and hence

$$HC_i(T_A(M)) = HC_i(A) \oplus HC_i(A \xrightarrow{i} T_A(M)) \tag{43}$$

To prove the theorem we show that

$$HC_i(A \xrightarrow{i} T_A(M)) = \oplus_{n=0}^{\infty} H_i(Z_{n+1}; S^n(A, M))) \tag{44}$$

Clearly

$$T_A(V_\bullet)/(A + [T_A(V_\bullet), T_A(V_\bullet)] + \mathrm{Im}(1 - t_n) = \oplus_{n=0}^{\infty} V_\bullet^{\otimes^{(n+1)}} / ([T_A(V_\bullet), T_A(V_\bullet)] + \mathrm{Im}(1 - t_n)). \tag{45}$$

Then we have the following isomorphism:

$$\oplus_{n=0}^{\infty} P_\bullet^{\otimes^{(n+1)}} / ([T_A(V_\bullet), T_A(V_\bullet)] + \mathrm{Im}(1 - t_n)) \approx \oplus_{n=0}^{\infty} A \otimes_{A \otimes A^{op}} P_\bullet^{\otimes^{(n+1)}} / \mathrm{Im}(1 - t_n). \tag{46}$$

The homology of the chain complex :

$$\oplus_{n=0}^{\infty} A \otimes_{A \otimes A^{op}} V_\bullet^{\otimes^{(n+1)}} / \mathrm{Im}(1 - t_n)$$

is equivalent to

$$\oplus_{n=0}^{\infty} H_i(Z_{n+1}; S^n(A, M)))$$

From 43 and 44 the proof is completed.

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